RATIONAL CANONICAL FORM OF POLYPHASE MATRICES WITH APPLICATIONS TO DESIGNING PARAUNITARY FILTER BANKS

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ABSTRACT

In this paper we consider the rational canonical form of arbitrary polyphase matrices and use it to derive a simple implementation of paraunitary filter banks (PUFBs) based on a cascade of elementary building blocks. Furthermore, this decomposition is shown to be easily extendable to include a large class of perfect reconstruction filter banks (PRFBs) and can be especially useful for deriving the initial condition of PUFB design algorithms.

Index Terms— rational canonical form, polyphase matrices, paraunitary filter banks

1. INTRODUCTION

The polyphase matrix of an M-channel causal finite impulse response (FIR) filter bank may be written as

$$\mathbf{H}(z)=\sum_{k=0}^{K}\mathbf{H}_k z^{-k}$$

where $z = re^{j\omega}$ and $\mathbf{H}_k$ is a scalar M-by-M matrix. The matrix $\mathbf{H}(z)$ resides within the ring of M-by-M matrices with Laurent polynomial entries, denoted by $\mathbf{M}(N, \mathbf{C}[z, z^{-1}])$, with $\mathbf{C}$ the field of complex numbers. The rational canonical form of this matrix is not guaranteed to exist within the Laurent polynomial ring $\mathbf{C}[z,z^{-1}]$ since this ring is not a field. In this paper we will show that the rational canonical form of $\mathbf{H}(z)$ does indeed exist as another matrix in the ring $\mathbf{M}(N, \mathbf{C}[z, z^{-1}])$. If $\mathbf{H}(z)$ corresponds to the polyphase matrix of a paraunitary filter bank (PUFB), then the algorithm for computing the rational canonical form illustrated in this paper may be used to implement $\mathbf{H}(z)$ as a particularly simple product of elementary building blocks. Furthermore, this new decomposition of $\mathbf{H}(z)$ may be used to represent a large class of perfect reconstruction filter banks (PRFBs) as well and can be used to provide initial conditions for PUFB design algorithms.

2. EXISTENCE OF RATIONAL CANONICAL FORM

The rational canonical form of a polyphase matrix $\mathbf{H}(z)$ over the ring $\mathbf{C}[z, z^{-1}]$ exists as a matrix over the field $\mathbf{K}$. $\mathbf{K}$ denotes the field of rational functions $p(z)/q(z)$, $q(z)\neq 0$, with coefficients in $\mathbf{C}$. The field $\mathbf{K}$ is the field of fractions of the ring $\mathbf{C}[z, z^{-1}]$. This field is also denoted $\mathbf{C}(z)$. Any matrix $\mathbf{H}(z)$ in $\mathbf{M}(N, \mathbf{C}[z, z^{-1}])$ is similar to a matrix in $\mathbf{M}(N, \mathbf{K})$ in rational canonical form. In other words, there exists an invertible M-by-M matrix $\mathbf{P}(z)$ such that $\mathbf{P}(z)^{-1}\mathbf{H}(z)\mathbf{P}(z)$ is in rational canonical form. A matrix $\mathbf{R}(z)$ in rational canonical form is a matrix such as

$$\mathbf{R}(z)\equiv\begin{bmatrix}\mathbf{C}_1(z) & 0 & \cdots & 0\\ 0 & \mathbf{C}_2(z) & \vdots & \vdots\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & \mathbf{C}_M(z)\end{bmatrix}$$

where

$$\mathbf{C}_i(z)\equiv\begin{bmatrix}0 & 0 & \cdots & -p_0(z)/q_0(z)\\ 1 & 0 & \cdots & -p_1(z)/q_1(z)\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 1 & -p_{n-1}(z)/q_{n-1}(z)\end{bmatrix}\in\mathbf{M}(N,\mathbf{K}).$$

The matrix $\mathbf{C}_i(z)$ is the companion matrix corresponding to the invariant factor $a_i(x)$ of $\mathbf{H}(z)$. The invariant factors $a_i(x)$ may be made unique by requiring that $p_i(z)$ and $q_i(z)$ are monic polynomials.

The characteristic polynomial of $\mathbf{H}(z)$ is a polynomial in the indeterminant x with coefficients in the Laurent polynomial ring $\mathbf{C}[z, z^{-1}]$, namely $c(x) = \det[x\mathbf{I} - \mathbf{H}(z)]$. The invariant factors of $\mathbf{R}(z)$ are the monic factors of $c(x)$ [1]. The invariant factors of $\mathbf{H}(z)$ and $\mathbf{R}(z)$ are the same, namely the monic polynomials $a_i(x)$.

The important contribution of this section is the following claim.

Claim: The rational canonical form of the polyphase matrix $\mathbf{H}(z)$ is a matrix with entries in the ring $\mathbf{C}[z, z^{-1}]$. In other words, it is an element in the ring $\mathbf{M}(N, \mathbf{C}[z, z^{-1}])$.

Proof: Since $\mathbf{C}$ is a field, the ring $\mathbf{C}[z, z^{-1}]$ is a unique factorization domain (UFD). The field $\mathbf{K} = \mathbf{C}(z)$ is the field of fractions of $\mathbf{C}[z, z^{-1}]$. The characteristic polynomial $c(x)$ is in the ring $\mathbf{C}[z, z^{-1}][x]$. By Gauss' Lemma, since $c(x)$ can be factored in $\mathbf{C}(z)[x]$, then it is reducible in $\mathbf{C}[z, z^{-1}][x]$. Consequently, the invariant factors of a matrix $\mathbf{H}(z)$ in $\mathbf{M}(N, \mathbf{C}[z, z^{-1}])$ are polynomials in the ring $\mathbf{C}[z, z^{-1}]$ and therefore the rational canonical form $\mathbf{R}(z)$ is a matrix in the ring $\mathbf{M}(N, \mathbf{C}[z, z^{-1}])$.

3. AN ALTERNATIVE FORM FOR R(Z)

Claim: The rational canonical form $\mathbf{R}(z)$ of a polyphase matrix $\mathbf{H}(z)$ can be written as a Laurent polynomial with matrix coefficients. In other words,

$$\mathbf{R}(z)\equiv\begin{bmatrix}\mathbf{C}_1(z) & 0 & \cdots & 0\\ 0 & \mathbf{C}_2(z) & \vdots & \vdots\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & \mathbf{C}_M(z)\end{bmatrix}$$
$$=\mathbf{R}_0+\mathbf{R}_1 z^{-1}+\mathbf{R}_2 z^{-2}+\cdots+\mathbf{R}_N z^{-N}.$$

Proof: The claim follows once one proves that the ring of matrices with Laurent polynomial entries is isomorphic to the ring of Laurent polynomials with matrix coefficients. It is sufficient to prove that [2],

$$\mathbf{M}(N,\mathbf{C}[x])\cong\mathbf{M}(N,\mathbf{C})[x]$$

which implies that

$$\mathbf{M}(N,\mathbf{C}[z,z^{-1}])\cong\mathbf{M}(N,\mathbf{C})[z,z^{-1}].$$

The proof is omitted for brevity.

4. COMPUTATION

To compute the rational canonical form $\mathbf{R}(z)$ of the M-by-M matrix $\mathbf{H}(z)$, one approach is to diagonalize the matrix $x\mathbf{I} - \mathbf{H}(z)$ [1]. The invariant factors of $\mathbf{H}(z)$ will then appear on the diagonal. The invariant factors should be monic and satisfy the divisibility condition $a_i(x)$ divides $a_{i+1}(x)$. The direct sum of the companion matrices associated with each invariant factor yields the matrix $\mathbf{R}(z)$ in rational canonical form. By keeping track of the row operations used to diagonalize $x\mathbf{I} - \mathbf{H}(z)$ one can also construct the matrix $\mathbf{P}(z)$ such that $\mathbf{P}(z)^{-1}\mathbf{H}(z)\mathbf{P}(z) = \mathbf{R}(z)$.

The following three elementary row and column operations can be used to diagonalize $x\mathbf{I} - \mathbf{H}(z)$:

1. interchange two rows or columns
2. add a multiple in $\mathbf{K}[x]$ of one row or column to another, e.g. add $p(x)$ times the jth row to the ith row
3. multiply any row or column by a unit in $\mathbf{K}[x]$, i.e. by a nonzero element in $\mathbf{K}$.

The matrix $\mathbf{P}(z)$ can also be computed systematically [1]. First, let $d_1,\ldots,d_m$ denote the degrees of the monic nonconstant polynomials $a_1(x),\ldots,a_m(x)$ appearing on the diagonal. Begin with the matrix $\mathbf{S} = \mathbf{I}$, the identity matrix. For each row operation used to diagonalize $x\mathbf{I} - \mathbf{H}(z)$, change the matrix $\mathbf{S}$ as follows.

1. If the ith and jth rows were interchanged, then interchange the ith and jth columns of $\mathbf{S}$,
2. if $Row_i + p(x)Row_j \rightarrow Row_i$, then subtract the product of the matrix $p(\mathbf{H}(z))$ times the ith column of $\mathbf{S}$ from the jth column of $\mathbf{S}$, i.e. $Col_j - p(\mathbf{H}(z))Col_i \rightarrow Col_j$,
3. if the ith row is multiplied by a unit, u, then divide the ith column of $\mathbf{S}$ by u.

Once the matrix $x\mathbf{I} - \mathbf{H}(z)$ has been diagonalized, the first $M - m$ columns of $\mathbf{S}$ will be zero. Then for each $i = 1,\ldots,m$ multiply the ith nonzero column of $\mathbf{S}$ successively by $\mathbf{H}(z)^0 = \mathbf{I}$, $\mathbf{H}(z)$, $\mathbf{H}(z)^2,\ldots, \mathbf{H}(z)^{d_i-1}$, where d_i is the degree of $a_i(x)$. Use the resulting column vectors in this order as the next d_i columns of a matrix $\mathbf{P}(z)$. Then $\mathbf{P}(z)^{-1}\mathbf{H}(z)\mathbf{P}(z) = \mathbf{R}(z)$.

5. APPLICATION TO PUFBS

In this section we consider the example of paraunitary filter banks. The analysis (or synthesis) polyphase matrix of a normalized PUFB satisfies the condition,

$$\mathbf{H}(z)\mathbf{H}(z)^H = \mathbf{I} \qquad \text{for } z = e^{j\omega}.$$

Furthermore, M-channel PUFBs may be decomposed into a product of elementary building blocks as in [3],

$$\mathbf{H}(\omega) = \mathbf{G}_L\mathbf{\Lambda}(\omega)\cdots\mathbf{G}_1\mathbf{\Lambda}(\omega)\mathbf{QJ} \qquad (1)$$

where L is the Smith-McMillan degree of $\mathbf{H}(\omega)$, $\mathbf{G}_k$ and $\mathbf{Q}$ are M-by-M orthogonal matrices, $\mathbf{\Lambda}(\omega) = \mathrm{diag}(\mathbf{I}, e^{-j\omega}\mathbf{I})$ for M even, and $\mathbf{J} = \mathrm{diag}(\pm1,\ldots,\pm1)$. Each matrix $\mathbf{G}_k$ and $\mathbf{Q}$ can be written as the product of $\frac{1}{2}M(M - 1)$ Givens rotation matrices in the sequence,

$$\mathbf{G}_k = \{\mathbf{B}_{M-2,M-1}\}\cdots\{\mathbf{B}_{1,M-1}\cdots\mathbf{B}_{12}\}\{\mathbf{B}_{0,M-1}\cdots\mathbf{B}_{01}\}. \qquad (2)$$

The matrix $\mathbf{B}_{ij}$ corresponds to a Givens rotation matrix with $\cos(\theta_n)$ in the ith row and ith column, $\sin(\theta_n)$ in position (i, j), $-\sin(\theta_n)$ in position (j, i), and $\cos(\theta_n)$ in position (j, j) with $1 \le n \le \frac{1}{2}M(M - 1)$.

5.1. Case $L = 1$, $M = 2$

For a simple example, let $M = 2$ and $L = 1$ in (1) above. Without loss of generality set $\mathbf{QJ} = \mathbf{I}$. Then the polyphase matrix is

$$\mathbf{H}(z) = \mathbf{G}_1\mathbf{\Lambda}(z) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}. \qquad (3)$$

Table 1 lists the steps to compute the rational canonical form of $\mathbf{H}(z)$ and the change of basis matrix $\mathbf{P}(z)$. In this table, R_i denotes the ith row, and C_j denotes the jth column.

Table 1. Sample Rational Canonical Form Computations

Steps to compute $\mathbf{R}(z)$	Steps to compute $\mathbf{P}(z)$
Step 1: Form the matrix $x\mathbf{I} - \mathbf{H}(z)$	**Step 1**: Form the matrix $\mathbf{S} = \mathbf{I}$
Step 2: $-\sin(\theta)R_1 \rightarrow R_1$, $(x - \cos(\theta))R_2 \rightarrow R_2$	**Step 2**: $-(1/\sin(\theta))C_1 \rightarrow C_1$, $(\mathbf{H}(z) - \cos(\theta)\mathbf{I})C_1 + C_2 \rightarrow C_1$
Step 3: $-R_1 + R_2 \rightarrow R_2$	**Step 3**: $\sin(\theta)C_1 \rightarrow C_1$
Step 4: $(1/\sin(\theta))R_1 \rightarrow R_1$	**Step 4**: $(z/\sin(\theta))C_2 \rightarrow C_2$, $-C_1 \rightarrow C_1$
Step 5: $z^{-1}\sin(\theta)R_2 \rightarrow R_2$, $(x^2 - x\cos(\theta)(1+z^{-1}) + z^{-1})R_1 \rightarrow R_1$	**Step 5**: $C_1 - (\mathbf{H}(z)^2 - \cos(\theta)\mathbf{H}(z)(1 + z^{-1}) + z^{-1}\mathbf{I})C_2 \rightarrow C_2$
Step 6: $-R_1 \rightarrow R_1$	**Step 6**: $(\mathbf{H}(z) - \cos(\theta)\mathbf{I})C_1 \rightarrow C_1$, $\sin(\theta)z^{-1}C_2 \rightarrow C_2$
Step 7: $R_1 + R_2 \rightarrow R_1$	**Step 7**: $(\mathbf{H}(z)^2 - \cos(\theta)\mathbf{H}(z)(1 + z^{-1}) + z^{-1}\mathbf{I})C_1 \rightarrow C_1$
Step 8: $(1/(x - \cos(\theta))R_1 \rightarrow R_1$, $(1/(z^{-1}\sin(\theta))R_2 \rightarrow R_2$	**Step 8**: Now, $\mathbf{S} = \begin{bmatrix} 0 & 0 \\ 0 & -2z^{-1}\sin(\theta)^2 \end{bmatrix}$ Also, $m = 1$, $d_1 = 2$.
Step 9: $(1/(x^2 - x\cos(\theta)(1 + z^{-1}) + z^{-1})R_1 \rightarrow R_1$	**Step 9**: Form the matrix, $\mathbf{P}(z) = [\mathbf{I}C_2 \quad \mathbf{H}(z)C_2]$

The result of these computations is

$$\mathbf{R}(z) = \begin{bmatrix} 0 & -z^{-1} \\ 1 & \cos(\theta)(1+z^{-1}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & \cos(\theta) \end{bmatrix} + z^{-1}\begin{bmatrix} 0 & -1 \\ 0 & \cos(\theta) \end{bmatrix},$$

$$\mathbf{P}(z) = -2\sin(\theta)^2 z^{-1}\begin{bmatrix} 0 & z^{-1}\sin(\theta) \\ 1 & z^{-1}\cos(\theta) \end{bmatrix}, \qquad (4)$$

$$\mathbf{P}(z)^{-1} = -\tfrac{1}{2}z\csc(\theta)^2\begin{bmatrix} -\cot(\theta) & 1 \\ z\csc(\theta) & 0 \end{bmatrix}.$$

Furthermore, $\mathbf{P}(z)^{-1}\mathbf{H}(z)\mathbf{P}(z) = \mathbf{R}(z)$.

Using the rational canonical form of $\mathbf{H}(z)$, one can write $\mathbf{H}(z) = \mathbf{H}_1(z) + z^{-1}\mathbf{H}_2(z)$ where $\mathbf{H}_1(z)$ and $\mathbf{H}_2(z)$ are rank one matrices defined as

$$\mathbf{H}_1(z) = \begin{bmatrix} \cos(\theta) \\ \cos(\theta)\cot(\theta) \end{bmatrix}\begin{bmatrix} (1 - z^{-1}) & z^{-1}\tan(\theta) \end{bmatrix},$$

$$\mathbf{H}_2(z) = \begin{bmatrix} \cos(\theta) \\ \csc(\theta)(\cos(\theta)^2 - z) \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Now for $M = 2$, the decomposition of PUFBs given in (1) can be rewritten as,

$$\mathbf{H}(z) = \prod_{i=1}^{L}\left(\mathbf{H}_{1i}(z) + z^{-1}\mathbf{H}_{2i}(z)\right).$$

6. SIMPLE IMPLEMENTATION OF PUFBS

Using the decomposition $\mathbf{P}_i(z)\mathbf{R}_i(z)\mathbf{P}_i(z)^{-1} = \mathbf{G}_i\mathbf{\Lambda}(z)$ for each building block of a *2*-by-*2* polyphase matrix in (1), a two channel PUFB may be constructed as a cascade of the lattice sections shown in Fig. 1. Here v_1, v_2 are the input signals and u_1, u_2 are the output signals. This lattice structure lends itself easily to an implementation in hardware.

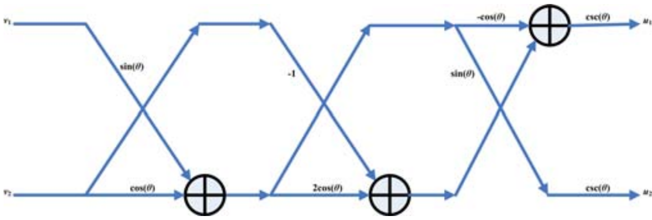

Fig. 1. Lattice Implementation of PUFB

Similar lattice structures can be derived for PUFBs with greater than M = 2 channels using (2), the rational canonical form for (3) and the rational canonical form for a scalar Givens rotation matrix. Using the same computational algorithm described in Section 4, a scalar Givens rotation matrix can be written in rational canonical form with $\mathbf{PRP}^{-1} = \mathbf{G}$ as,

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 2\cos(\theta) \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} 0 & \sin(\theta) \\ 1 & \cos(\theta) \end{bmatrix},$$
$$\mathbf{P}^{-1} = \begin{bmatrix} -\cot(\theta) & 1 \\ \csc(\theta) & 0 \end{bmatrix}. \qquad (5)$$

6.1. Case $M = 3$

Using (2), a 3-by-3 paraunitary polyphase matrix may be written as,

$$\mathbf{H}(z) = \mathbf{B}_{12}(\theta_1)\mathbf{B}_{02}(\theta_2)\mathbf{B}_{01}(\theta_3)\mathbf{\Lambda}(z),$$

with $\mathbf{\Lambda}(z)$ = diag(1, z^{-1}, z^{-1}). A lattice implementation can be constructed from the following matrices derived using (4) and (5),

$$\mathbf{B}_{12}(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \sin(\theta_1) \\ 0 & 1 & \cos(\theta_1) \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 2\cos(\theta_1) \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cot(\theta_1) & 1 \\ 0 & \csc(\theta_1) & 0 \end{bmatrix},$$

$$\mathbf{B}_{02}(\theta_2) = \begin{bmatrix} 0 & 0 & \sin(\theta_2) \\ 0 & 1 & 0 \\ 1 & 0 & \cos(\theta_2) \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2\cos(\theta_2) \end{bmatrix}\begin{bmatrix} -\cot(\theta_2) & 0 & 1 \\ 0 & 1 & 0 \\ \csc(\theta_2) & 0 & 0 \end{bmatrix},$$

$$\mathbf{B}_{01}(\theta_3)\mathbf{\Lambda}(z) = \begin{bmatrix} 0 & \sin(\theta_3) & 0 \\ 1 & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}\begin{bmatrix} 0 & -z^{-1} & 0 \\ 1 & \cos(\theta_3)(1+z^{-1}) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\cot(\theta_3) & 1 & 0 \\ z\csc(\theta_3) & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similar matrices and lattice structures may be derived using (2), (4), and (5) for paraunitary polyphase matrices with M = 4, or larger. Those examples will not be presented here to conserve space.

7. PERFECT RECONSTRUCTION FILTER BANKS

Consider the case M = 2 again. A perfect reconstruction filter bank satisfies the property that $\mathbf{H}(z)\mathbf{E}(z) = cz^{-m}\mathbf{I}$, where $\mathbf{H}(z)$ is the analysis polyphase matrix and $\mathbf{E}(z)$ is the synthesis polyphase matrix, $c \neq 0$ and m is an integer. A PUFB is the special case where $\mathbf{E}(z) = \mathbf{H}^{H}(z)$. A perfect reconstruction filter bank may be constructed by parameterizing the rational canonical form $\mathbf{R}(z)$ and the conjugation matrix $\mathbf{P}(z)$ independently using one angle for each. For example, the system with

$$\mathbf{H}(z) = \begin{bmatrix} 0 & \sin(\theta_1) \\ 1 & \cos(\theta_1) \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2\cos(\theta_2) \end{bmatrix}\begin{bmatrix} -\cos(\theta_1) & \sin(\theta_1) \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}\csc(\theta_1)$$

$$\mathbf{E}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}\begin{bmatrix} 0 & \sin(\theta_1) \\ 1 & \cos(\theta_1) \end{bmatrix}\begin{bmatrix} 2\cos(\theta_2) & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -\cos(\theta_1) & \sin(\theta_1) \\ 1 & 0 \end{bmatrix}\csc(\theta_1) \qquad (6)$$

is PR since $\mathbf{E}(z) = \mathbf{H}(z)^{-1}$. Note that for the special case when $\theta_1 = \theta_2$, $\mathbf{H}(z)$ is paraunitary. Therefore, this set of PRFBs constructed using two parameters for every building block includes the set of PUFBs as a proper subset.

8. APPLICATION TO PUFB DESIGN

Since the space of PRFBs is larger than and includes the space of PUFBs, one can design a PUFB by searching over the space of PRFBs for an optimal solution and then choosing the paraunitary filter bank closest to it for an initial condition in a strictly PUFB search. The advantage of this approach is that if the initial condition of a strictly PUFB search is closer to the optimal solution, then the more likely it is that a nonlinear optimization program will settle on the global solution instead of a local minimum. Consider the following PUFB design algorithm which minimizes the Mean Square Error (MSE), ζ, between a desired polyphase matrix, $\mathbf{D}(\omega)$, and an approximation $\mathbf{H}(\omega)$.

8.1. PUFB Design Algorithm

Step 1. Form the MSE objective function,

$$\mathrm{MSE}(\mathbf{D}(\omega),\mathbf{P}(\omega)) = \zeta = \tfrac{1}{2\pi}\int_0^{2\pi} W(\omega)\|\mathbf{D}(\omega) - \mathbf{P}(\omega)\|_F^2\, d\omega$$

where $\|\ .\ \|_F$ denotes the Frobenius norm, $W(\omega)$ is a scalar weighting function set equal to one, $\mathbf{P}(\omega)$ is an M-by-M perfect reconstruction polyphase matrix as described in Section 7, and $\mathbf{D}(\omega)$ is the desired or ideal polyphase matrix.

Step 2. With the angles θ_i as free parameters, minimize ζ over all $\mathbf{P}(\omega)$ using a nonlinear optimization program such as the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm. Call the optimal solution, $\mathbf{P}^*(\omega)$.

Step 3. Compute the paraunitary polyphase matrix $\mathbf{H}'(\omega)$, parameterized as in (1) and (2), which minimizes MSE($\mathbf{P}^*(\omega)$, $\mathbf{H}(\omega)$) over all $\mathbf{H}(\omega)$.

Step 4. Use $\mathbf{H}'(\omega)$ as the initial starting point in a nonlinear optimization program which minimizes MSE($\mathbf{D}(\omega)$, $\mathbf{H}(\omega)$) over all $\mathbf{H}(\omega)$. Call this optimal solution $\mathbf{H}^*(\omega)$. Denote MSE($\mathbf{D}(\omega)$, $\mathbf{H}^*(\omega)$) by MSE_0.

The advantage of using the proposed PUFB design algorithm is that it satisfies the following property.

Claim: Define MSE_1 = MSE($\mathbf{D}(\omega)$, $\mathbf{H}'(\omega)$). Using the above filter bank design algorithm,

$$\mathrm{MSE}_1 \leq \mathrm{MSE}_0 + \varepsilon,$$
$$\mathrm{MSE}(\mathbf{H}'(\omega), \mathbf{H}^*(\omega)) \leq 2 \cdot \mathrm{MSE}_0 + \varepsilon$$

where ε is a small nonnegative constant.

Proof: Since the MSE is a norm on the space of polyphase matrices, by the triangle inequality,

$$\tfrac{1}{2\pi}\int_0^{2\pi}\|\mathbf{D}(\omega) - \mathbf{H}'(\omega)\|_F^2\, d\omega \leq \tfrac{1}{2\pi}\int_0^{2\pi}\|\mathbf{D}(\omega) - \mathbf{P}^*(\omega)\|_F^2\, d\omega +$$
$$\tfrac{1}{2\pi}\int_0^{2\pi}\|\mathbf{P}^*(\omega) - \mathbf{H}'(\omega)\|_F^2\, d\omega.$$

Since $\mathbf{P}^*(\omega)$ is the optimal solution over a larger set of matrices than $\mathbf{H}^*(\omega)$,

$$\tfrac{1}{2\pi}\int_0^{2\pi}\|\mathbf{D}(\omega) - \mathbf{P}^*(\omega)\|_F^2\, d\omega \leq \tfrac{1}{2\pi}\int_0^{2\pi}\|\mathbf{D}(\omega) - \mathbf{H}^*(\omega)\|_F^2\, d\omega.$$

As a result,

$$\frac{1}{2\pi}\int_0^{2\pi}\left\|\mathbf{D}(\omega)-\mathbf{H}'(\omega)\right\|_F^2 d\omega \le \frac{1}{2\pi}\int_0^{2\pi}\left\|\mathbf{D}(\omega)-\mathbf{H}^*(\omega)\right\|_F^2 d\omega + \frac{1}{2\pi}\int_0^{2\pi}\left\|\mathbf{P}^*(\omega)-\mathbf{H}'(\omega)\right\|_F^2 d\omega,$$

or in other words, $\text{MSE}_1 \le \text{MSE}_0 + \varepsilon$. The proof that $\text{MSE}(\mathbf{H}'(\omega), \mathbf{H}^*(\omega)) \le 2\cdot\text{MSE}_0 + \varepsilon$ is similar.

8.2. Case $M = 4$

For $M = 4$, a PR polyphase matrix, $\mathbf{P}(z)$, which can be used to design a PUFB using the previously described algorithm is constructed as,

$$\mathbf{P}(z) = \mathbf{G_1}(\theta_1,\ldots,\theta_{12})\mathbf{\Lambda}(z) = \mathbf{T}_{23}(\theta_1,\theta_2)\mathbf{T}_{13}(\theta_3,\theta_4)\mathbf{T}_{12}(\theta_5,\theta_6)\mathbf{T}_{03}(\theta_7,\theta_8)\mathbf{T}_{02}(\theta_9,\theta_{10})\mathbf{T}_{01}(\theta_{11},\theta_{12})\mathbf{\Lambda}(z).$$

The inverse of $\mathbf{P}(z)$ is given by,

$$\mathbf{P}(z)^{-1} = \mathbf{\Lambda}(z)^H\mathbf{B}_{01}(\theta_{11},\theta_{12})\mathbf{B}_{02}(\theta_9,\theta_{10})\mathbf{B}_{03}(\theta_7,\theta_8)\mathbf{B}_{12}(\theta_5,\theta_6)\mathbf{B}_{13}(\theta_3,\theta_4)\mathbf{B}_{23}(\theta_1,\theta_2)$$

with the matrices $\mathbf{B}_{ij}$ as in (5) and (2) except with two angle parameters as in (6). The system so described is PR since $\mathbf{P}(z)\mathbf{P}(z)^{-1} = \mathbf{I}$. The matrix $\mathbf{T}_{23}$ is,

$$\mathbf{T_{23}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sin(\theta_1) \\ 0 & 0 & 1 & \cos(\theta_1) \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2\cos(\theta_2) & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\cot(\theta_1) & 1 \\ 0 & 0 & \csc(\theta_1) & 0 \end{bmatrix}.$$

The matrices $\mathbf{T}_{13}$, $\mathbf{T}_{12}$, $\mathbf{T}_{03}$, $\mathbf{T}_{02}$, $\mathbf{T}_{01}$ are constructed similarly using two angle parameters for each. Note that if $\theta_i = \theta_{i+1}$ for i odd, then $\mathbf{P}(z)$ is paraunitary.

9. RESULTS

The PUFB design algorithm was used to design a finite impulse response PU approximation to an ideal principal component filter bank (PCFB). PCFBs are described in detail in [4], [5], and [6]. A PCFB is the solution to the problem of finding optimal Q-by-M and M-by-Q analysis and synthesis polyphase matrices, with $Q \le M$, such that the time-averaged mean squared error between the vector input to the filter bank and the vector output is minimized. An ideal PCFB corresponding to an infinite order PU filter bank has channel filters with brick-wall responses. This ideal filter bank can be approximated using a FIR PU filter bank by minimizing the mean squared Frobenius norm error between the desired polyphase matrix of the ideal PCFB, $\mathbf{D}(\omega)$, and the FIR PU synthesis polyphase matrix, $\mathbf{H}(\omega)$. PCFBs are an interesting design example because they are optimal for maximizing coding gain and minimizing mean-squared error in the presence of quantization noise. It has been proven that they are also optimal for any concave function of the subband variance vector [6].

Table 2 lists the performance of the PUFB design algorithm and compares it to another elegant design algorithm described by Tkacenko in [4]. Tkacenko generously made available the MATLAB code to duplicate his results on the Internet. The PR polyphase matrix was populated with incrementally more two-parameter building blocks. As the table shows, the MSE of the optimal PR solution decreased monotonically with greater degrees of freedom, until the number of free parameters was too great for the algorithm to converge. A PU approximation to the optimal PR solution with the largest number of free parameters was then used as the initial condition for a search over PU matrices to arrive at the final filter bank solution. The length of the channel filters in the final PUFB was eight taps.

Table 2. Performance of PUFB Design Algorithm

Algorithm	MSE($\mathbf{D}(\omega)$, $\mathbf{H}^*(\omega)$)
Tkacenko PU solution	1.7024
PU search with random start value	1.5747
PR matrix with 14 angles	1.5578
PR matrix with 16 angles	1.5549
PR matrix with 18 angles	1.5539
PR matrix with 20 angles	1.5415
PR matrix with 22 angles	1.5395
PR matrix with 24 angles	3.6687 (algorithm did not converge)
PU search with PU initial condition	**1.5741**

As Table 2 shows, the lowest MSE for a PUFB was attained with the PU initial condition. Figure 2 illustrates the frequency response of one of the optimal channel filters (red), and overlays the ideal filter response (blue dotted line), with the Tkacenko solution (black).

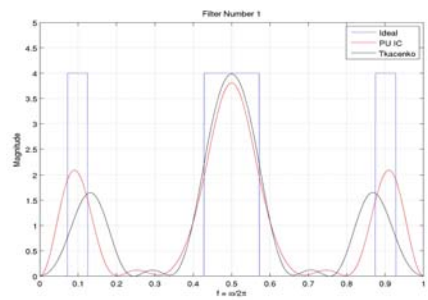

Fig 2. Channel Filter One

As the curves show, the derived filter is an excellent approximation to the ideal brick-wall filter with high frequency selectivity in the narrow bandpass regions.

10. CONCLUSIONS

In this paper we showed that the rational canonical form of a polyphase matrix exists in the ring $\mathbf{M}(N, \mathbf{C}[z, z^{-1}])$ and can be written as a matrix in the ring $\mathbf{M}(N, \mathbf{C})[z, z^{-1}]$. The factorization of paraunitary polyphase matrices into a similarity transformation of the rational canonical form makes possible a very simple lattice implementation of the filter bank. Also, this decomposition lends itself easily to a representation of a large class of perfect reconstruction filter banks and can be used to find an initial condition for PUFB design algorithms that ultimately yields better MSE performance.

11. REFERENCES

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