Algebraic Construction of MIMO Radar Waveforms

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***Abstract*—This paper describes a method for constructing waveforms such that the individual radars in a spatial Multiple Input Multiple Output (MIMO) radar system can simultaneously transmit pulse compression sequences yet avoid interference with each other. The waveform design algorithm is based on the matrix representations of finite abelian groups.**

I. Background

Multiple Input Multiple Output (MIMO) radar configurations are an active area of research in the radar community [1-4]. MIMO radar can encompass several different system configurations. A spatial MIMO radar is a system that transmits independent waveforms from multiple spatially separated transmit antennas, and receives the signals on multiple spatially separated receive antennas [1]. The individual radars in the MIMO system work together to detect and track targets and because the system has improved angular diversity as compared to a single radar, target detection performance is increased. Since the waveform transmitted by any radar in the group may be received by any other radar, it is desirable that the waveforms transmitted by each radar are mutually orthogonal to avoid interference. However, it is difficult to construct orthogonal pulse compression codes that satisfy this requirement. Pulse compression waveforms are highly desirable in radar applications and pulse compression is a commonly used technique to provide maximum energy on target using a long transmit pulse without sacrificing the high range resolution associated with a short pulse [5]. This paper describes a procedure for embedding a given pulse compression sequence within a larger set of orthogonal waveforms such that each radar in the MIMO system can transmit the same pulse compression code and avoid interference with its neighbors.

II. Overview

For radars, pulse compression codes are highly effective transmit waveforms because they efficiently utilize a radar's time to put maximum energy on target. Although desirable, not every radar in a spatial MIMO group can transmit the same pulse compression sequence at the same time due to the interference this configuration would create. To avoid mutual interference, it is possible to embed a pulse compression sequence $c(n)$ of length N within a larger set of orthogonal sequences $d_j(n)$ by forming the Kronecker product,

$$d_j(n) = c(n) \otimes f_j(n), \tag{1}$$

where each vector $f_j(n)$ constitutes a row of a unitary matrix $\mathbf{F}$. The vectors $f_j(n)$ could be assigned to each radar in the MIMO group to act as unique keys that encode and decode the radar's transmissions.

Assume each radar in the MIMO group is assigned a unique key of length M equal to a row of the matrix $\mathbf{F}$. The key $f_j(n)$ is used to encode the pulse compression sequence $c(n)$ and to transmit a sequence $d_j(n)$ of total length MN. Since the vectors $f_j(n)$ are orthogonal, the vectors $d_j(n)$ are also orthogonal and the radars avoid interference with each other. On receive, each radar's key can be used to recover the original sequence $c(n)$ for pulse compression. The strategy described is similar to cellular communications using the Code Division Multiple Access (CDMA) standard. All the users in a cell occupy the same spectrum, but each user transmits using a unique code to avoid interference.

The remainder of this paper is organized as follows. Section III provides the necessary background in group representation theory to present the waveform design algorithm. Section IV describes the approach for deriving the unitary matrix $\mathbf{F}$ for a group of M MIMO radars and Section V provides some illustrative examples of waveform processing and performance.

III. Matrix Representation of Algebraic Groups

This section will briefly present some necessary background on algebraic groups and their representations. Most of the tutorial material can be found in references 6 through 9.

A group G is defined as a set of elements together with a binary operation *. Any two elements g and h of G may be combined to form another element of G, written as $g*h$ or abbreviated as gh. The group operation must satisfy the following axioms:

- associativity, meaning that for all g, h, and k in G, $(gh)k = g(hk)$;

- there exists an identity element e in G such that for all g in G, $eg = ge = g$;
- for all g in G, there exists an inverse element g^{-1} in G such that $gg^{-1} = g^{-1}g = e$.

The operation used to combine elements of G shall be referred to as the product operation on G.

A group G is said to be abelian if $gh = hg$ for all g and h in G. Abelian groups will be used extensively to construct the codes presented in this paper. Another important class of groups that will be used extensively is the cyclic groups. A group G is cyclic if the entire group can be generated by a single element g in G, in which case we write $G = < g >$. When the group operation is multiplication, G will be equal to all integer powers of g, that is, $< g > = \{g^n: n \text{ in } \boldsymbol{Z}\}$. In additive notation, G is cyclic if $G = \{ng: n \text{ in } \boldsymbol{Z}\}$. The group of integers modulo n under addition, denoted $\boldsymbol{Z}/n\boldsymbol{Z}$, is an example of a cyclic group. It is generated by the element $1 + n\boldsymbol{Z}$. In fact, every finite cyclic group is isomorphic to $\boldsymbol{Z}/n\boldsymbol{Z}$. The set of the nth roots of unity in the field of complex numbers is a cyclic group of cardinality (or order) n and is denoted C_n. For example, if $a = e^{2\pi j/n}$, then $C_n = \{1, a, a^2,\ldots, a^{n-1}\}$ and $a^n = 1$. Thus, C_n is isomorphic to $\boldsymbol{Z}/n\boldsymbol{Z}$. Any group of prime order is cyclic and therefore isomorphic to C_p. Any group of order p^2, p prime, is abelian.

If G is a group, then a subset H of G is called a subgroup of G, if H is itself a group under the product operation inherited from G. The notation $H \leq G$ is used to denote that H is a subgroup of G. A subgroup N of a group G is said to be a normal subgroup of G if $g^{-1}Ng = N$ for all g in G. Note the notation $g^{-1}Ng$ signifies the set $\{g^{-1}ng: n \text{ in } N\}$.

If G is a group of order $p^\alpha m$, where α is an integer and p is a prime that does not divide m, then a subgroup of order p^α is called a Sylow p-subgroup of G and is written as N_p. The number of Sylow p-subgroups of G will be denoted by n_p and is of the form $1 + kp$, k an integer. In other words,

$$n_p \equiv 1 \bmod p\ . \tag{2}$$

Furthermore, n_p divides m. For any finite group G and p a prime number dividing the order of G, there exists a Sylow p-subgroup of G. A useful fact is that if $n_p = 1$, then N_p is normal in G.

One method for constructing a new group from existing groups is to form the direct product. Given two groups G and H, the direct product $G \times H$ is the set of ordered pairs $\{(g, h)$: g in G, h in $H\}$. The product operation on $G \times H$ is defined by $(g, h)(g', h') = (gg', hh')$ for all g, g' in G and all h, h' in H.

For the case of multiple groups, $G_1,\ldots,G_M$, the direct product is

$$G_1 \times \cdots \times G_M = \{(g_1,\ldots,g_M) : g_i \in G_i \text{ for } 1 \leq i \leq M\}. \tag{3}$$

The product operation is given by

$$(g_1,\ldots,g_M)(g_1',\ldots,g_M') = (g_1g_1',\ldots,g_Mg_M'). \tag{4}$$

If all the groups G_i are finite, then the order of $G_1 \times \cdots \times G_M$ is $|G_1| \cdots |G_M|$, where $|G_i|$ denotes the order of G_i.

Every cyclic group of composite order is decomposable into the direct product of cyclic groups whose orders are powers of distinct prime numbers. In other words, if n is any positive integer, then

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}\text{, and } C_n \cong C_{p_1^{\alpha_1}} \times C_{p_2^{\alpha_2}} \times \cdots \times C_{p_k^{\alpha_k}}\ , \tag{5}$$

where the p_i are prime numbers. Also, the direct product $\boldsymbol{Z}/n\boldsymbol{Z} \times \boldsymbol{Z}/m\boldsymbol{Z}$ is isomorphic to $\boldsymbol{Z}/mn\boldsymbol{Z}$, if and only if, n and m are coprime, meaning the greatest common divisor of n and m is one. For example, $\boldsymbol{Z}/12\boldsymbol{Z}$ is isomorphic to $\boldsymbol{Z}/4\boldsymbol{Z} \times \boldsymbol{Z}/3\boldsymbol{Z}$ but is not isomorphic to $\boldsymbol{Z}/6\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$. If n is the product of distinct primes, then up to isomorphism the only abelian group of order n is the cyclic group C_n of order n.

Functions between groups which preserve the group structure will be a central ingredient in the subsequent development. If G and H are groups, then a homomorphism from G to H is a function θ: $G \rightarrow H$ such that $\theta(g_1g_2) = \theta(g_1)\theta(g_2)$ for all g_1, g_2 in G. An invertible homomorphism is an isomorphism.

A representation of a group G is a homomorphism from G into a group of invertible matrices. Let GL(n, F) denote the group of n-by-n invertible matrices with entries in the field F, which is taken to be the real or complex numbers. A representation of G over F is a homomorphism ρ from G to GL(n, F) for some n. The degree of ρ is the integer n. Thus if ρ is a function from G to GL(n, F), then ρ is a representation if and only if $\rho(gh) = \rho(g)\rho(h)$ for all g, h in G. The representations of abelian groups will form the basis for the encoding algorithm to be presented later in this paper.

Let V be a vector space over the field F and let G be a group. Then V is an FG-module if a multiplication vg (v in V, g in G) is defined satisfying the following conditions for all u, v in V, λ in F, and g, h in G:

- vg is in V;
- $v(gh) = (vg)h$;
- $v1 = v$;
- $(\lambda v)g = \lambda(vg)$;
- $(u + v)g = ug + vg$.

The letters F and G in the word FG-module indicate that V is a vector space over F and that G is the group from which the elements g are taken to form the products vg, with v in V.

If V is an FG-module, then a subset W of V is said to be an FG-submodule of V if W is a subspace and wg is in W for all w in W and all g in G. Thus an FG-submodule of V is a subspace which is also an FG-module.

There exists a bijection between any FG-module V and a pair (V, ρ) with V as a vector space over F and ρ: $G \rightarrow$ GL(V) a representation. In other words, specifying a representation ρ: $G \rightarrow$ GL(V) on a vector space V over the field F is equivalent to specifying an FG-module V. Under this correspondence, it is said that the module V affords the representation ρ of G.

An FG-module V is said to be irreducible if it is non-zero and it has no FG-submodules apart from $\{0\}$ and V. If V has an FG-submodule W with W not equal to $\{0\}$ or V, then V is reducible. A representation ρ is called irreducible or reducible according to whether the FG-module affording it has the corresponding property. Reducible representations are those with a corresponding matrix representation whose matrices are in block triangular form. In other words, if ρ is a reducible representation of a finite group G over F of degree n, then

$$\rho(g) = \begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix} \tag{6}$$

for g in G and for some matrices **X**, **Y**, and **Z** where **X** is size k-by-k with $0 < k < n$.

By the Fundamental Theorem of Abelian Groups, every finite abelian group is isomorphic to a direct product of cyclic groups. Thus if A is an abelian group, then

$$A \cong C_{n_1} \times C_{n_2} \times \cdots \times C_{n_r} \tag{7}$$

where $n_1,\ldots,n_r$ are positive integers. The irreducible representations of all finite abelian groups have degree one. By determining the irreducible representations of direct products of the form (7), one can describe the irreducible representations of finite abelian groups.

Let the element c_i be a generator for C_{n_i}, with $1 \le i \le r$, and let

$$G = C_{n_1} \times C_{n_2} \times \cdots \times C_{n_r}. \tag{8}$$

Write $g_i = (1,\ldots,c_i,\ldots,1)$ with c_i in the ith position. Then,

$$G = \langle g_1,\ldots,g_r \rangle, \tag{9}$$

with $g_i^{n_i} = 1$ and $g_ig_j = g_jg_i$ for all i, j. The bracket notation $< g_1,\ldots,g_r >$ denotes the group generated by the elements $g_1,\ldots,g_r$. This group is equal to the set of products,

$$\langle g_1,\ldots,g_r \rangle = \left\{ g_1^{i_1} g_2^{i_2} \cdots g_r^{i_r} \;\middle|\; i_j \text{ an integer for all } j \right\}. \tag{10}$$

Now let ρ: $G \to \mathrm{GL}(n, F)$ be an irreducible representation of G over the complex numbers. Then $n = 1$, so for $1 \le i \le r$, there exist complex scalars λ_i such that

$$\rho(g_i) = \lambda_i. \tag{11}$$

But since g_i has order n_i, meaning $g_i^{n_i} = 1$, then $\lambda_i^{n_i} = 1$ and λ_i is an n_ith root of unity. Furthermore, for any g in G,

$$g = g_1^{i_1} \cdots g_r^{i_r} \tag{12}$$

for some integers $i_1,\ldots,i_r$, and

$$\rho(g) = \rho\left(g_1^{i_1} \cdots g_r^{i_r}\right) = \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_r^{i_r}. \tag{13}$$

Thus, the scalars λ_i determine ρ. Any representation ρ of G that satisfies (13) is irreducible and has degree 1. Conversely, given any n_ith roots of unity λ_i for $1 \le i \le r$, the function

$$g_1^{i_1} \cdots g_r^{i_r} \to \lambda_1^{i_1} \cdots \lambda_r^{i_r} \tag{14}$$

is a representation of G. There are $|G| = n_1n_2\cdots n_r$ such representations and no two of them are equivalent.

As an example, consider the cyclic group,

$$G = C_n = \left\langle a : a^n = 1 \right\rangle, \tag{15}$$

and let $W = e^{2\pi j/n}$. The n irreducible representations of G over the complex numbers are ρ_m, $0 \le m \le n-1$, where

$$\rho_m\left(a^k\right) = W^{mk} = e^{2\pi jmk/n}, \quad 0 \le k \le n-1. \tag{16}$$

To elaborate further on the aspects of representation theory essential to this paper, consider the conjugacy classes of a finite group G. Given x, y in G, x is conjugate to y if

$$y = g^{-1}xg \tag{17}$$

for some g in G. The set of all elements conjugate to x in G is

$$x^G = \left\{ g^{-1}xg : g \in G \right\}, \tag{18}$$

and is called the conjugacy class of x in G. Two distinct conjugacy classes have no elements in common. A class function on G is a function ψ from G to the complex numbers such that $\psi(x) = \psi(y)$ whenever x and y are conjugate elements of G. In other words, class functions are constant on conjugacy classes.

A. Characters

Let ρ: $G \to \mathrm{GL}(n, F)$ be a representation of the finite group G. With each n-by-n matrix $\rho(g)$, g in G, is associated the trace of the matrix. The character χ_ρ of the representation ρ shall mean the function from G to the complex numbers such that $\chi_\rho(g) = \mathrm{tr}\{\rho(g)\}$, for all g in G. Recall that the trace is the sum of the diagonal elements of a matrix and it is a class function since it does not depend on the choice of basis for the matrix. Characters encapsulate a great deal of information about a representation.

A simple or irreducible character of G is the character of an irreducible representation of G and only a finite number of simple characters of G exist. The number of irreducible characters of G is equal to the number of conjugacy classes of G. All the irreducible characters of a finite group are distinct.

Likewise, a reducible character is the character of a reducible representation. The characters of reducible representations are linear combinations of the simple characters with nonnegative integer coefficients. If ψ and χ are characters, then so is their product $\psi\chi$.

Equivalent representations have the same character. Two representations $\rho: G \rightarrow \mathrm{GL}(n, F)$ and $\sigma: G \rightarrow \mathrm{GL}(m, F)$ of G are equivalent if n equals m and there exists an invertible n-by-n matrix $\mathbf{T}$ such that for all g in G,

$$\sigma(g) = \mathbf{T}^{-1}\rho(g)\mathbf{T} . \tag{19}$$

Every character is constant on conjugacy classes of G, which is a consequence of the fact that characters of representations are class functions.

The degree of a character is the degree of any representation affording it. Characters of degree one are called linear characters and are irreducible. The values of characters of degree one are the nth roots of unity. For finite abelian groups, irreducible complex representations are equal to their characters, since all the irreducible representations of finite abelian groups are one-dimensional scalars constructed as in (13). For arbitrary groups, finding all the irreducible characters is a difficult process.

If e denotes the identity element of any group G, then a representation ρ of G of degree n maps e to the n-by-n identity matrix. Therefore, $\chi_\rho(e)$ equals the degree of ρ. Furthermore, for g in G,

$$\chi_\rho\left(g^{-1}\right) = \overline{\chi_\rho(g)} \tag{20}$$

where the overbar denotes complex conjugation.

Since the space of functions from a group G to the complex numbers is a vector space, one can define an inner product between characters. Let χ and ψ be characters of G. Then the inner product of χ and ψ is a real number given by

$$\langle \chi, \psi \rangle = \langle \psi, \chi \rangle = \frac{1}{|G|}\sum_{g \in G} \chi(g)\psi\left(g^{-1}\right). \tag{21}$$

This result can be simplified by assuming G has exactly m conjugacy classes with representatives $g_1,\ldots,g_m$. By definition, if

$$G = g_1^G \cup \cdots \cup g_m^G \tag{22}$$

where the conjugacy classes $g_1^G,\ldots,g_m^G$ are distinct, then the elements $g_1,\ldots,g_m$ are called representatives of the conjugacy classes of G. Also, define the centralizer of g in G, denoted by $C_G(g)$, to be the set

$$C_G(g) = \{x \in G : xg = gx\} = \{x \in G : x^{-1}gx = g\}. \tag{23}$$

In other words, the centralizer of g in G is the set of all elements in G that commute with g and is also a subgroup of G. A useful fact is that the size of the conjugacy class g^G is given by

$$\left|g^G\right| = \frac{|G|}{|C_G(g)|} . \tag{24}$$

By using (24) and the fact that any group is the disjoint union of its conjugacy classes, equation (21) can be rewritten as

$$\langle \chi, \psi \rangle = \sum_{i=1}^{m} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|} . \tag{25}$$

The irreducible characters $\chi_1,\ldots,\chi_k$ of a finite group G form an orthonormal set such that

$$\langle \chi_i, \chi_j \rangle = \delta_{ij} , \tag{26}$$

where δ_{ij} is the Kronecker delta, that is $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. If ψ is any character of G, then

$$\psi = \alpha_1\chi_1 + \cdots + \alpha_k\chi_k , \tag{27}$$

where

$$\alpha_i = \langle \psi, \chi_i \rangle \geq 0 \text{ and } \|\psi\|^2 = \langle \psi, \psi \rangle = \sum_{i=1}^{k} \alpha_i^2 . \tag{28}$$

A character has norm one if and only if it is irreducible.

B. Character Tables

Let $\chi_1,\ldots,\chi_k$ be the irreducible characters of G and let the elements $g_1,\ldots,g_k$ be representatives of the k conjugacy classes of G. The k-by-k matrix whose ijth-entry is $\chi_i(g_j)$ for $1 \leq i, j \leq k$ is called the character table of G. Thus, the rows of the character table are indexed by the irreducible characters of G, and the columns are indexed by the conjugacy classes. Note that the ordering of the conjugacy classes is arbitrary and that the character table of G is an invertible matrix.

The orthogonality relation (26) among the irreducible characters $\chi_1,\ldots,\chi_k$ of G can be expressed in terms of the rows of the character table as

$$\sum_{i=1}^{k} \frac{\chi_r(g_i)\overline{\chi_s(g_i)}}{|C_G(g_i)|} = \delta_{rs} , \tag{29}$$

for any values of r and s between 1 and k.

A similar orthogonality relation exists between the columns of the character table. Specifically,

$$\sum_{i=1}^{k} \chi_i(g_r)\overline{\chi_i(g_s)} = \delta_{rs} \left| C_G(g_r) \right| . \tag{30}$$

Note the value of equation (30) is $|C_G(g_r)|$ if g_r and g_s are conjugate in G, and is zero otherwise.

One can adjust the entries $\chi_i(g_j)$ in a k-by-k character table to create another k-by-k matrix **F** which is unitary. Letting

$$[\mathbf{F}]_{ij} = \frac{\chi_i(g_j)}{\left|C_G(g_j)\right|^{1/2}} \qquad (31)$$

implies that $\mathbf{F}^H\mathbf{F} = \mathbf{F}\mathbf{F}^H = \mathbf{I}_k$.

A very useful construction in this paper will be the character table of a direct product of groups as in (3). Consider two groups G and H. Let $\chi_1,\ldots,\chi_k$ be the distinct irreducible characters of G and let $\psi_1,\ldots,\psi_m$ be the distinct irreducible characters of H. Then $G \times H$ has precisely km distinct irreducible characters

$$\chi_i \times \psi_j, \quad 1 \le i \le k, 1 \le j \le m. \qquad (32)$$

As a result, the character table of the direct product $G \times H$ is equal to the tensor product of the character tables for G and H. The character table of an abelian group G of order n will be an n-by-n matrix, since G has n irreducible representations and each element in G is in its own conjugacy class (because $g^{-1}xg = y$ implies $x = y$ for all g in G).

IV. Waveform Design Algorithm

By exploiting the structure present in matrix representations of finite abelian groups, a set of orthogonal vectors can be constructed to form the rows of the matrix **F**, which form the set of unique keys assigned to each radar. The following algorithm describes the procedure.

1. Choose the number M of orthogonal vectors to create.
2. Determine all abelian groups of order, or cardinality, M. Choose one group G for constructing **F**. Determining all the abelian groups of a given order is generally not a trivial task. However, it is straightforward to determine how many distinct, non-isomorphic abelian groups of a given order exist by using the following reasoning. Any integer $M > 1$ can be uniquely factored into a distinct product of prime powers as

$$M = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}. \qquad (33)$$

   If q_i is the number of partitions of α_i, then the number of distinct, non-isomorphic abelian groups of order M equals $q_1q_2\cdots q_k$. For example, consider $M = 16 = 2^4$. The partitions of 4 are 1+1+1+1, 3+1, 2+2, 2+1+1, and 4. So there exist 5 non-isomorphic abelian groups of order 16.
3. Write the chosen abelian group G as the direct product of k cyclic groups, as in (7).
4. Determine the matrix representation, or character table $\boldsymbol{K}_j$, $1 \le j \le k$, for each cyclic factor.
5. Determine the character table of G, denoted **M**, by taking the tensor product of the character tables of each cyclic factor, that is,

$$\mathbf{M} = \mathbf{K}_1 \otimes \cdots \otimes \mathbf{K}_k. \qquad (34)$$

6. Transform the matrix **M** into a unitary matrix **F** using (31). The centralizer of an element g in an abelian group G is equal to the entire group. Therefore, for abelian groups, normalize **M** by sqrt(|G|) to create a unitary matrix.
7. Extract each row of the unitary matrix **F** to construct a set of mutually orthogonal keys.

A. Design Examples

The design examples presented in this section are selected to illustrate each step of the proposed waveform design algorithm. The first example illustrates Step 2 of the algorithm. Example 2 provides a listing of abelian groups of order less than 23. This list may be used to complete Step 3 of the design algorithm. Example 3 illustrates the result of applying all the steps in the design algorithm.

1) Example 1. $M = p^3$, p prime

The abelian groups of order p^3 are C_{p^3}, $C_{p^2} \times C_p$ and $C_p \times C_p \times C_p$. Each group has a different character table.

2) Example 2. $M \le 22$

Table 1 lists all the non-isomorphic abelian groups of order less than 23.

TABLE I. Abelian Groups of Small Order

Order	Abelian Groups
1	C_1
2	C_2
3	C_3
4	C_4, $C_2 \times C_2$
5	C_5
6	$C_6 \approx C_3 \times C_2$
7	C_7
8	C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$
9	C_9, $C_3 \times C_3$
10	$C_{10} \approx C_5 \times C_2$
11	C_{11}
12	$C_{12} \approx C_4 \times C_3$, $C_6 \times C_2$
13	C_{13}
14	$C_{14} \approx C_7 \times C_2$
15	$C_{15} \approx C_5 \times C_3$
16	C_{16}, $C_8 \times C_2$, $C_4 \times C_4$, $C_4 \times C_2 \times C_2$, $C_2 \times C_2 \times C_2 \times C_2$
17	C_{17}
18	C_{18}, $C_6 \times C_3$
19	C_{19}
20	C_{20}, $C_{10} \times C_2$
21	$C_{21} \approx C_7 \times C_3$
22	$C_{22} \approx C_{11} \times C_2$

3) Example 3. C_2, $C_2 \times C_2$, and C_4

Consider C_2, the cyclic group of order 2. The group C_2 has two conjugacy classes and two irreducible characters. The character table of C_2 is the 2-by-2 matrix in bold type below.

TABLE II. CHARACTER TABLE OF C_2

Representative of Conjugacy Class	1	*g*
χ_1	**1**	**1**
χ_2	**1**	**-1**

The group $C_2 \times C_2$ equals the set

$$C_2 \times C_2 = \left\{ (1,1), (g,1), (1,h), (g,h) : g^2 = h^2 = 1 \right\}. \quad (35)$$

The character table of $C_2 \times C_2$ is provided by Table III.

TABLE III. CHARACTER TABLE OF $C_2 \times C_2$

Conjugacy Class	(1, 1)	(*g*, 1)	(1, *h*)	(*g*, *h*)
χ_1	**1**	**1**	**1**	**1**
χ_2	**1**	**1**	**-1**	**-1**
χ_3	**1**	**-1**	**1**	**-1**
χ_4	**1**	**-1**	**-1**	**1**

The group $C_4 = < g: g^4 = 1 >$ and its character table is provided by Table IV.

TABLE IV. CHARACTER TABLE OF C_4

Conjugacy Class	1	*g*	g^2	g^3
χ_1	**1**	**1**	**1**	**1**
χ_2	**1**	**i**	**-1**	**-i**
χ_3	**1**	**-1**	**1**	**-1**
χ_4	**1**	**-i**	**-1**	**i**

The important observation from this example is that the character table of C_4 is not equal to the character table of $C_2 \times C_2$, which is not equal to the tensor product of the character table of C_2 with itself, because gcd(2, 2) ≠ 1.

4) Example 4. $M = 175$

In this case, $|G| = 175 = 5^2 \cdot 7$. Notice n_5 must divide 7 and $n_5 \equiv 1 \bmod 5$ which implies $n_5 = 1$ and N_5 is normal. Similarly, n_7 divides 25 and $n_7 \equiv 1 \bmod 7$, so $n_7 = 1$, which implies N_7 is normal. Then, $|N_5| = 5^2$ which implies N_5 is abelian since it has order p^2, p prime. Also, $|N_7| = 7$ implies N_7 is cyclic since it has prime order. Thus, $G = N_5 \times N_7$ because if the Sylow subgroups of a finite group G are normal, then G is the direct product of its Sylow subgroups. Furthermore, $G = N_5 \times N_7$ is abelian since N_5 and N_7 are abelian. Thus, $G = C_5 \times C_5 \times C_7$ or $G = C_{25} \times C_7$ by the Fundamental Theorem of Abelian Groups.

5) Example 5. $M = 10$

Since $10 = 5 \cdot 2$ and $\gcd(5, 2) = 1$, $C_{10} = C_5 \times C_2$, which implies that the character table for C_{10} is the tensor product of the character tables for C_5 and C_2. The character table for C_2 is the matrix {1, 1; 1, -1}. The character table for C_5 is the matrix,

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \xi & \xi^2 & \overline{\xi^2} & \overline{\xi} \\ 1 & \overline{\xi} & \overline{\xi^2} & \xi^2 & \xi \\ 1 & \xi^2 & \overline{\xi} & \xi & \overline{\xi^2} \\ 1 & \overline{\xi^2} & \xi & \overline{\xi} & \xi^2 \end{bmatrix}, \quad (36)$$

where $\xi = \exp(2\pi j/5)$ and the overbar denotes complex conjugation. The matrix **F** follows after dividing **M** by sqrt(10).

V. RECEIVER PROCESSING

Figure 1 provides a time-domain illustration of a portion of a typical transmit sequence for the case where $M = 10$ and the length of the pulse compression sequence $c(n)$ is 512 bits. Figure 2 illustrates the power spectrum of this waveform.

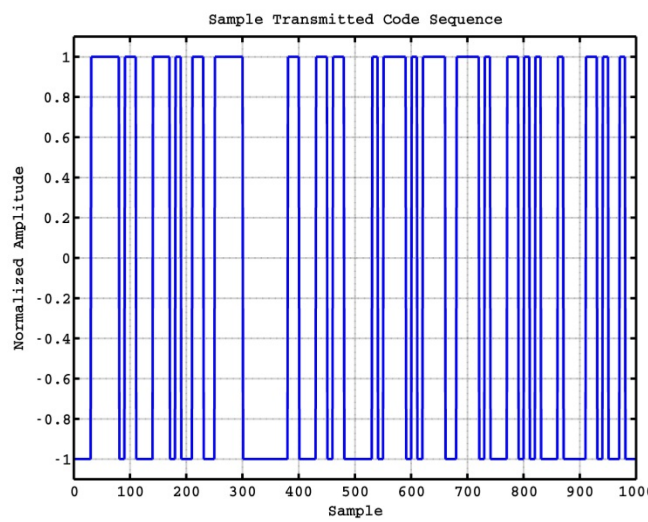

Figure 1. Sample Transmit Sequence

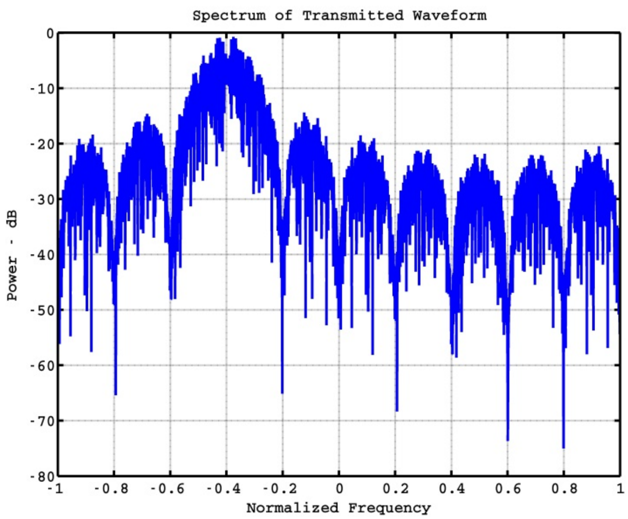

Figure 2. Transmit Signal Spectrum

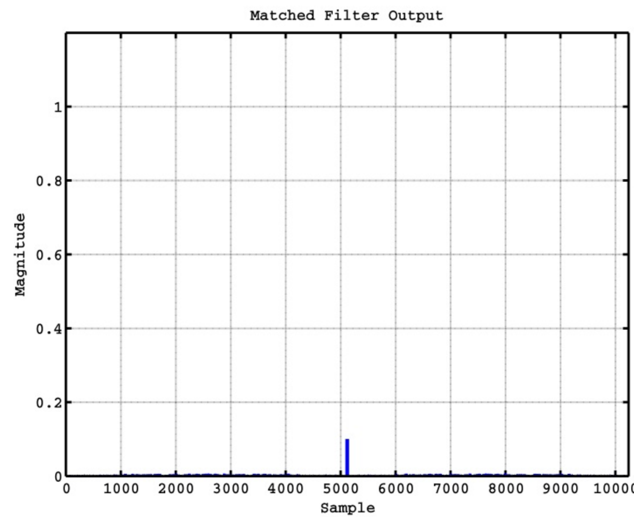

Figure 4. Matched Filter Output: $k = 10, j = 1$

A. *Matched Filter Receiver*

The standard radar receiver design is a matched filter. In this architecture, the digitized receive waveform $x_k(n)$ at the kth radar is convolved with a length-MN, time-reversed replica of the transmitted waveform $d_k(n)$. The result of this operation is shown in Fig. 3 for the case $k = 10$.

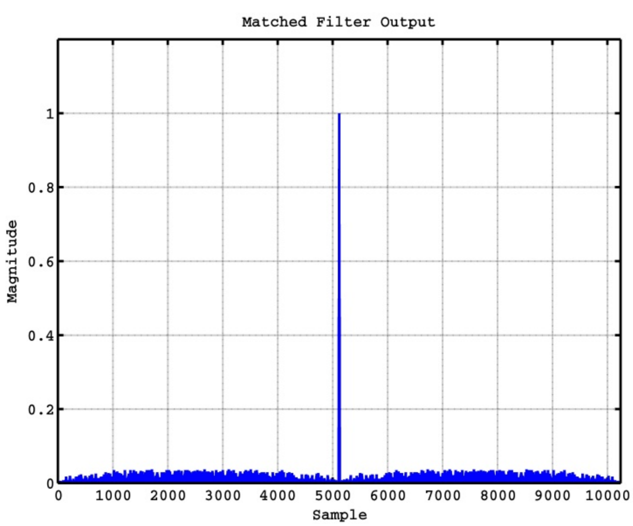

Figure 3. Matched Filter Output

The drawback to this approach is that the cross-correlation between mismatched transmit and receive waveforms is high enough to create false targets. Figure 4 displays the output of the matched filter receiver when a waveform transmitted by the kth radar is processed by the jth radar, with $k = 10$ and $j = 1$. Figure 5 illustrates the matched filter output when $k = 10$ and $j = 9$. In both cases, the distinct peak in the receiver's output could be mistaken for a target.

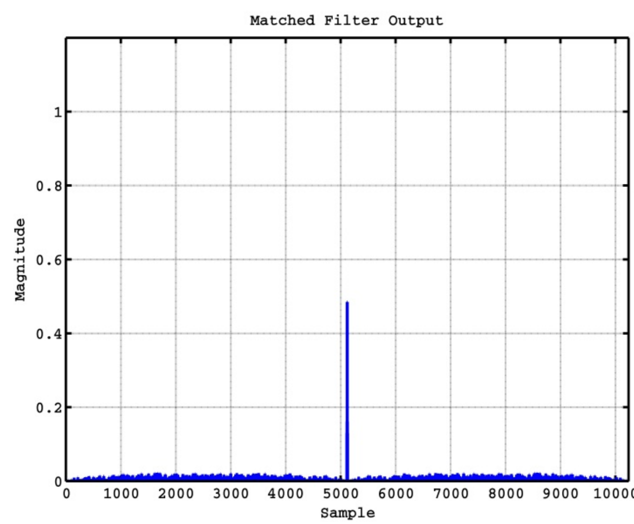

Figure 5. Matched Filter Output: $k = 10, j = 9$

To mitigate the potential interference issues created by using matched filter processing, a two-stage receiver design is proposed in the next section.

B. *Two-Stage Receiver*

In this implementation, the receive processing at the jth radar required to extract $c(n)$ from the transmitted sequence $d_j(n)$ consists of two stages. After the received signal has been digitized to form the discrete-time signal $x_j(n)$, the first stage of processing consists of forming the dot product between a vector of blocked signal samples of length M and the jth key $f_j(n)$. Recall that the jth key is assigned to the jth radar. The second stage of processing convolves the output of the first stage with a time-reversed replica of the pulse compression sequence $c(n)$. The two stages of receive processing are summarized by the following pseudocode, where $r = 1, 2, \ldots, N$ and the 1-by-N vector $\mathbf{v}$ denotes the output of the first stage:

$$\mathbf{f}_j = \left[\, f_j(1) \quad f_j(2) \quad \cdots \quad f_j(M) \,\right]^T,$$
$$\mathbf{x}_r = \left[\, x(1+(r-1)M) \quad x(2+(r-1)M) \quad \cdots \quad x(rM) \,\right]^T,$$
$$\mathbf{c} = \left[\, c(N) \quad c(N-1) \quad \cdots \quad c(1) \,\right],$$

$$\text{STAGE 1:} \quad v(r) = \mathbf{f}_j^{\,H}\mathbf{x}_r, \quad \mathbf{v} = \left[\, v(1) \quad v(2) \quad \cdots \quad v(N) \,\right],$$

$$\text{STAGE 2:} \quad \mathbf{y} = \mathbf{v} * \mathbf{c}.$$

When using this receiver design, the output of the first stage will be identically zero if the transmitted sequence from radar j is processed using any key f_k with $k \neq j$. For the case when $j = k$, the output of both stages will be the nicely compressed radar pulse shown in the time domain in Figure 6. A potential drawback to this receiver design is that the output of Stage 1 will be identically zero for $k \neq j$, only if the jth key $\mathbf{f}_j$ and the blocked signal samples $\mathbf{x}_r$ are perfectly aligned. This issue will be addressed further in future work.

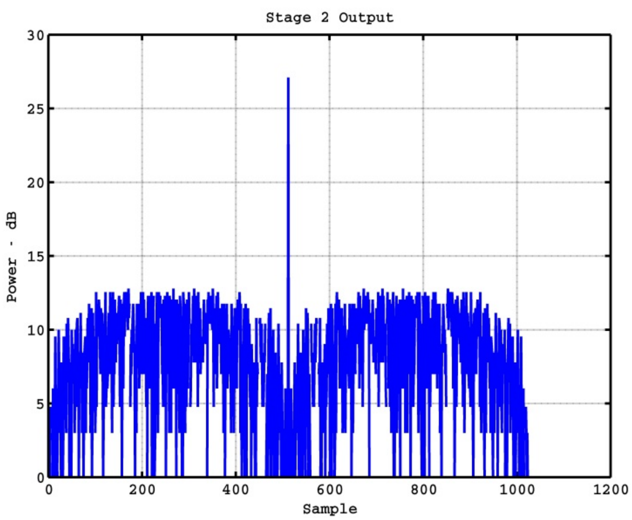

Figure 6. Two-Stage Receiver Output

C. Doppler Performance

As mentioned previously, the output of the first stage is identically zero if the kth radar attempts to process the waveform transmitted by the jth radar, which is a highly desirable property. However, if there is a Doppler shift imposed on the jth waveform, then the residue at the output of Stage 1 at the kth radar will start to increase. This effect was simulated by applying a linear phase ramp across the jth transmit sequence of length MN and then by processing it using the kth key. Figure 7 shows how the maximum residue at the output of Stage 1 steadily increases with larger Doppler shifts for the case where $j = 10$ and $k = 5$. The Doppler tolerance of the proposed waveforms will be addressed in greater detail in future research.

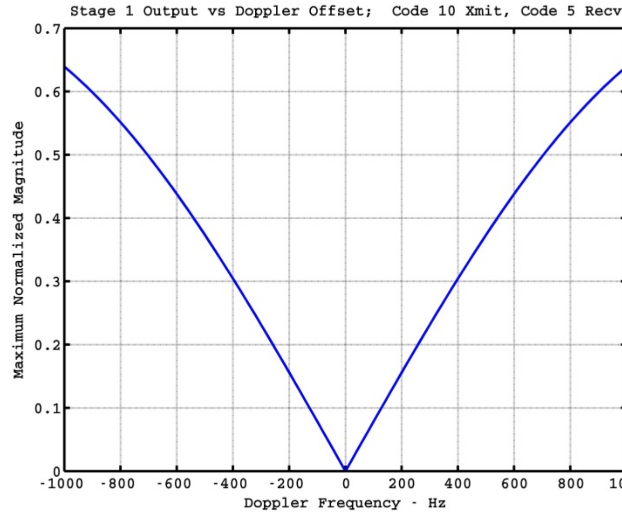

Figure 7. Waveform Doppler Performance

VI. Conclusions

In this paper, a new approach is presented for creating orthogonal transmit code sequences that preclude interference between multiple radars operating in a spatial MIMO environment. The approach allows each radar in the MIMO group to transmit and compress the same pulse compression code, at the expense of a long duration transmit waveform. Furthermore, a notional receiver design is presented, and a discussion of the Doppler tolerance of the proposed waveforms suggests good performance in the presence of target Doppler shifts.

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